SIAM J. COMPUT.
Vol. 5, No. 2, June 1976

RANKING ALGORITHMS: THE SYMMETRIES AND COLORATIONS OF THE n-CUBE*

JAY P. FILLMORE AND S. G. WILLIAMSON†

Abstract. This paper discusses determination of the ranks, in lexicographic lists, of the symmetries and proper symmetries of the n-cube and of its colorations by r colors, up to these symmetries.

Key words. combinatorial algorithms, ranking algorithms, serial numbering schemes, n-cube

1. Introduction. Classically, "enumerative combinatorics" has been concerned with counting rather than listing sets of combinatorial structures. A systematic discussion of basic questions on enumeration from the latter point of view was initiated by D. H. Lehmer in *The machine tools of combinatorics* [1]: a set of combinatorial structures should be analyzed by first giving an algorithm for its generation, and hence linearly ordering it, and second giving a ranking or "serial numbering" function, an easily computable function giving the rank or position of each structure within the linear order. Carrying out such an analysis is generally more difficult than merely determining the cardinality of a set of structures; its importance is amply indicated by Lehmer. A knowledge of the ranking function is essential for the effective use of an algorithm, as the exponential size of the outputs of many algorithms prevents their entire execution.

This paper deals with a classical problem, coloring the n-cube up to symmetries, from Lehmer's point of view. We present the ranking functions for four algorithms concerning the n-cube. A listing is:

1. the group of all symmetries,
2. the group of proper symmetries,
3. the colorations by r colors up to all symmetries,
4. the coloration by r colors up to proper symmetries.

In each case, the algorithm is the listing of the structures in a suitable lexicographic order.

2. The groups of the n-cube. Let G and H be groups acting on the sets A and B, respectively. G acts on the Cartesian product $A \times B$ via the first factor, $g \cdot (a, b) = (g \cdot a, b)$, a in A, etc.; the set H^A of all functions f from A to H is a group under pointwise operations and it acts on $A \times B$ by $f \cdot (a, b) = (a, f(a) \cdot b)$. Together G and H^A generate the wreath product, the semidirect product $G \cdot H^A$, in which H^A is normal, and which acts on $A \times B$.

The set of $2n$ faces of the n-cube is the Cartesian product of the set $A = \{1, 2, \cdots, n\}$ of n axes and a set $B = \{1, -1\}$ of two elements which distinguish faces associated with an axis; that is, the centers of the faces are $\pm e_i$, $e_1 = (1, 0, \cdots, 0)$, etc. The group of all $n!2^n$ symmetries of the n-cube is the wreath product $G \cdot H^A$, where G is the group of all permutations of A and H that of B. Denoting by σ an element of G and by $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$, $\varepsilon_i = \pm 1$, a function in H^A, $(\sigma, (\varepsilon_1, \cdots, \varepsilon_n))$ is the symmetry $e_i \to \varepsilon_{\sigma^{-1}(i)} e_{\sigma^{-1}(i)}$.

* Received by the editors August 22, 1974, and in revised form June 1, 1975.

† Department of Mathematics, University of California, San Diego, La Jolla, California 92037.

The research of the second author supported by the National Science Foundation under Grant GJ43333.

List the elements of G and H^A in lexicographic order, and those of $G \cdot H^A$ in the product lexicographic order.

The lexicographic rank g, $0 \leqq g < n!$, of the permutation

$$\begin{pmatrix} 1 & 2 & \cdots & n \\ a_1 & a_2 & \cdots & a_n \end{pmatrix} = a_1 a_2 \cdots a_n$$

is

$$g = a_1' \cdot (n-1)! + a_2' \cdot (n-2)! + \cdots + a_{n-1}' \cdot 1!$$

where a_i', $0 \leqq a_i \leqq n-i$, is the rank of a_i in $a_i a_{i+1} \cdots a_n$ [1, p. 20].

The rank h, $0 \leqq h < 2^n$, of the function $(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ is

$$h = \varepsilon_1' \cdot 2^{n-1} + \varepsilon_2' \cdot 2^{n-2} + \cdots + \varepsilon_{n-1}' \cdot 2 + \varepsilon_n',$$

where $\varepsilon_i' = 0$ or 1 as $\varepsilon_i = 1$ or -1.

The rank N, $0 \leqq N < n!2^n$, of the symmetry $(a_1 \cdots a_n, (\varepsilon_1, \cdots, \varepsilon_n))$ is

$$N = g \cdot 2^n + h.$$

For example: the 118th symmetry of the 4-cube has rank $N = 117$.

$$117 = 7 \cdot 2^4 + 5,$$

so $g = 7$, $h = 5$;

$$7 = 1 \cdot 3! + 0 \cdot 2! + 1 \cdot 1!,$$

so $a_1' a_2' a_3' = 101$, $a_1 a_2 a_3 a_4 = 2143$;

$$5 = 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2 + 1 \cdot 1,$$

so $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (1, -1, 1, -1)$.

The 118th symmetry is

$$(2143, (1, -1, 1, -1)) \quad \text{or} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The group of all $n!2^{n-1}$ proper symmetries of the n-cube consists of those symmetries $(\sigma, (\varepsilon_1, \cdots, \varepsilon_n))$ for which σ and $(\varepsilon_1, \cdots, \varepsilon_n)$ have like parity $\varepsilon_1 \cdot \cdots \cdot \varepsilon_n$.

List the elements of G and H^A as before and pair those of like parity.

The parity of the element of G of rank g is $(-1)^{[(g+1)/2]}$. If we observe that G consists of n lists of permutations on $n-1$ symbols, each list prefixed by $1, 2, \cdots,$ or n, a proof is obtained by induction.

The parity of the element of H^A of rank h is $p(h) = \varepsilon_1 \cdot \cdots \cdot \varepsilon_n$, where $h = \varepsilon_1' \cdot 2^{n-1} + \varepsilon_2' \cdot 2^{n-2} + \cdots + \varepsilon_{n-1}' \cdot 2 + \varepsilon_n'$, with $\varepsilon_i' = 0$ as $\varepsilon_i = 1$ or -1. $p(h)$ may be defined recursively by:

$$p(0) = 1, \quad p(1) = -1 \quad \text{and} \quad p(h) = -p(h'),$$

where

$$h' = h - (\text{largest power of 2 not exceeding } h).$$

The pairs of ranks 01, 23, 45, $\cdots$ have one rank of each parity. It follows that the kth element of H^A of a given parity appears in the kth pair. Of the elements of parity 1, the one of rank k appears for $h = 2k + (1 - p(k))/2$; of the elements of parity -1, the one of rank k appears for $h = 2k + (1 + p(k))/2$.

The rank N, $0 \leqq N < n!2^{n-1}$, of the symmetry $(\sigma, (\varepsilon_1, \cdots, \varepsilon_n))$ in the lexicographic list of proper symmetries is

$$N = g \cdot 2^{n-1} + k,$$

where g is the rank of σ, $0 \leqq g < n!$ and k is the rank of $(\varepsilon_1, \cdots, \varepsilon_n)$ in the list of functions of the same parity as that of σ, $0 \leqq k < 2^{n-1}$.

For example, the 118th proper symmetry of the 4-cube has rank $N = 117$.

$$117 = 14 \cdot 2^3 + 5,$$

so $g = 14$, $k = 5$;

$$14 = 2 \cdot 3! + 1 \cdot 2! + 0 \cdot 1!,$$

so $a_1'a_2'a_3' = 210$, $a_1a_2a_3a_4 = 3214$.

3214 has parity $(-1)^{[(14+1)/2]} = -1$.

$$h = 2 \cdot 5 + \frac{1 + p(5)}{2} = 11,$$

$$11 = 1 \cdot 2^3 + 0 \cdot 2^2 + 1 \cdot 2 + 1 \cdot 1,$$

so $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) = (-1, 1, -1, -1)$.

The 118th proper symmetry is

$$(3214, (-1, 1, -1, -1)) \quad \text{or} \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Other algorithms for listing the elements of G and H^A lead similarly to the listing of the symmetries and proper symmetries of the n-cube. The method of adjacent marks [1, p. 22] lists the permutations in G with alternating parity. The subsets of A may be listed first by cardinality and then lexicographically within sets of equal cardinality. A ranking function for the latter may be based on the combinatorial representation of integers [1, p. 28].

3. The colorations of the n-cube. Let S be a set. H acts on the sets S^B of all functions f from B to S via the action on B, $(h \cdot f)(b) = f(h^{-1} \cdot b)$; its orbit space we denote S^B/H. Likewise, G acts on $(S^B/H)^A$ via the action on A with orbit space $(S^B/H)^A/G$, and $G \cdot H^A$ acts on $S^{A \times B}$ via the action on $A \times B$ with orbit space $S^{A \times B}/G \cdot H^A$. There is a well-known canonical bijection between these latter two orbit spaces, as made evident in the geometric examples below. (For a

proof, note that the canonical bijection $S^{A\times B}/G\cdot H^A=(S^B/H)^A/G$ is given by

$$f \bmod G\cdot H^A \to (\text{the function } a'\to f(a',\cdot) \bmod H) \bmod G.$$

Functions of two variables can be viewed as functions of the first variable with values in the set of functions of the second variable; $S^{A\times B}/G\cdot H^A=(S^B)^A/G\cdot H^A$. H^A is normal in $G\cdot H^A$, so its action may be factored out first and the action of $G\cdot H^A/H^A$ second; $(S^B)^A/G\cdot H^A=((S^B)^A/H^A)/(G\cdot H^A/H^A)$. The action of H^A on functions from A to S^B is one copy of the action of H on S^B for each element of A; $(S^B)^A/H^A=(S^B/H)^A$. Finally, $G\cdot H^A/H^A=G$ is clear.)

The set of all distinct (under the symmetries in $G\cdot H^A$) labelings of the cube $A\times B$ by elements of a set S is the orbit space $S^{A\times B}/G\cdot H^A=(S^B/H)^A/G$. That is, place all distinct (under H) labeled axes onto the cube in all distinct (under G) ways.

For example, when $n=3$ and $S=\{0,1\}$, the distinct labeled axes are 00, 01, and 11. The distinct colorations of the cube, ten in number, are

00	00	00	00	11	11
00	00	01	01	01	01
00	00	11	01	01	11
00	01	01	01	11	11
00	01	11	11	11	11

Note that these are the lexicographically nondecreasing sequences of length three on the symbols 00, 01, and 11.

In general, when $S=\{0,1,\cdots,r-1\}$ and $G\cdot H^A$ is the group of all symmetries of the n-cube, the distinct labeled axes S^B/H are described by symbols ij, $0\leqq i\leqq j<r$; since G consists of all permutations of A, the colorations of the n-cube up to all symmetries, $S^{A\times B}/G\cdot H^A=(S^B/H)^A/G$, are described by lexicographically nondecreasing sequences of length n on the symbols ij. There are $l=\binom{r}{2}+r$ symbols ij and $\binom{l+n-1}{n}$ nondecreasing sequences of length n on these l symbols; hence there are

$$\binom{\binom{r}{2}+r+n-1}{n}$$

colorations of the n-cube by r colors, up to all symmetries.

There is a bijection between nondecreasing sequences $x_1x_2\cdots x_n$, $0\leqq x_1\leqq x_2\leqq\cdots\leqq x_n<l$, and $(l-1)$- or n-element subsets of an $(n+l-1)$-element set:

$$x_1x_2\cdots x_n \to \overbrace{0\ 0\ 0\ \cdots\ 0}^{d_1}\ 1\ \overbrace{0\ 0\ \cdots\ 0}^{d_2}\ 1\ \cdots\ 1\ \overbrace{0\ 0\ \cdots\ 0}^{d_l}$$

$$0\ 1\ 2 \qquad\qquad n+l-1$$

where $d_x \geqq 0$ is the number of x's in $x_1x_2 \cdots x_n$. The subsets are specified by the strictly increasing sequence of the ranks of the $l-1$ 1's or the n 0's. When the sequences $x_1x_2 \cdots x_n$ are listed in lexicographic increasing order:

(i) the sequences of length n of the ranks of the 0's, read left to right, appear in lexicographic increasing order.

(ii) the sequences of length $l-1$ of the ranks of the 1's, read right to left, appear in colexicographic increasing order.

Every integer N, $0 \leqq N < \binom{c}{k}$, has a unique combinatorial representation

$$N = \binom{b_1}{1} + \binom{b_2}{2} + \cdots + \binom{b_k}{k},$$

with $0 \leqq b_1 < b_2 < \cdots < b_k < c$ [1, p. 28]; as N increases, all such strictly increasing sequences $b_1b_2 \cdots b_k$ are produced in colexicographic order (the last unequal entry is strictly larger). Similarly,

$$\binom{c}{k} - 1 - N = \binom{c-1-a_k}{1} + \binom{c-1-a_{k-1}}{2} + \cdots + \binom{c-1-a_1}{k}$$

produces all strictly increasing sequences $a_1a_2 \cdots a_k$ in lexicographic order [1, p. 28].

For example, the $l = \binom{r}{2} + r$ symbols ij, $0 \leqq i \leqq j < r$, are nondecreasing sequences of length 2 on $0, 1, 2, \cdots, r-1$. To ij corresponds

$$\overbrace{1\ 1\ 1\ \cdots\ 1}^{i}\ 0\ \overbrace{1\ 1\ 1\ \cdots\ 1}^{j-i}\ 0\ \overbrace{1\ 1\ 1\ \cdots\ 1}^{r-j-i}.$$
$$0\ 1\ 2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad r-1$$

The rank N of the strictly increasing sequence $a_1, a_2 = i, j+i$ is given by

$$\binom{r+1}{2} - 1 - N = \binom{r-j-1}{1} + \binom{r-i}{2}.$$

The colorations of the n-cube by r colors are ranked by ranking the nondecreasing sequences of length n on the l symbols ij, $0 \leqq i \leqq j < r$, lexicographically ordered.

For example, there are

$$\binom{\binom{10}{2} + 10 + 5 - 1}{5} = \binom{59}{5} = 5006386$$

colorations of the 5-cube by 10 colors. The 1000000th coloration has rank $N = 999999$.

$$\binom{59}{5} - 1 - 999999 = \binom{58-a_5}{1} + \binom{58-a_4}{2} + \binom{58-a_3}{3} + \binom{58-a_2}{4} + \binom{58-a_1}{5},$$

$$4006386 = \binom{0}{1} + \binom{30}{2} + \binom{37}{3} + \binom{47}{4} + \binom{56}{5},$$

so $a_1a_2a_3a_4a_5 = 2\ \ 11\ \ 21\ \ 28\ \ 58$.

To the strictly increasing sequence described by

$$1^2 \quad 0 \quad 1^8 \quad 0 \quad 1^9 \quad 0 \quad 1^6 \quad 0 \quad 1^{29} \quad 0$$

corresponds the nondecreasing sequence:

$$2 \quad 10 \quad 19 \quad 25 \quad 54.$$

These are the ranks of the symbols

$$02 \quad 11 \quad 22 \quad 28 \quad 99$$

in the list of 55 symbols ij, $0 \leqq i \leqq j < 10$. This is the 1000000th coloration of the 5-cube by 10 colors.

In the opposite direction, the ranks of

$$01 \quad 23 \quad 45 \quad 67 \quad 89$$

in the list of 55 symbols ij are

$$1 \quad 20 \quad 35 \quad 46 \quad 53.$$

To this corresponds the strictly increasing sequence described by

$$1 \quad 0 \quad 1^{19} \quad 0 \quad 1^{15} \quad 0 \quad 1^{11} \quad 0 \quad 1^{7} \quad 0 \quad 1,$$

so $a_1a_2a_3a_4a_5 = 1 \quad 21 \quad 37 \quad 49 \quad 57$;

$$\binom{59}{5} - 1 - N = \binom{58-57}{1} + \binom{58-49}{2} + \binom{58-37}{3} + \binom{58-21}{4} + \binom{58-1}{5},$$

$$5006386 - 1 - N = \binom{1}{1} + \binom{9}{2} + \binom{21}{3} + \binom{37}{4} + \binom{57}{5} = 4254518,$$

so $N = 751867$.

This coloration of the 5-cube by 10 colors is the 751868th.

Under the group of all symmetries of the n-cube, every coloration by r colors $0, 1, \cdots, r-1$ is described by a nondecreasing sequence of length n on the symbols ij, $0 \leqq i \leqq j < r$. Under the group of *proper* symmetries of the n-cube, every coloration is equivalent to one of the above *or* to a coloration obtained by interchanging the first i and j in a strictly increasing sequence of symbols ij, $0 \leqq i \leqq j < r$. For whenever a symbol ij with $i = j$ appears or a symbol ij is repeated, a reflection perpendicular to an axis or an interchange of two axes leaves the coloration unchanged. There are

$$\binom{\binom{r}{2}}{n}$$

strictly increasing sequences of length n on the symbols ij, $0 \leqq i < j < r$, lexicographically ordered; hence there are

$$\binom{\binom{r}{2} + r + n - 1}{n} + \binom{\binom{r}{2}}{n}$$

colorations of the n-cube by r colors, up to proper symmetries.

For the purpose of ranking these colorations, the lexicographic list of nondecreasing sequences on the symbols ij, $0 \leqq i \leqq j < r$, is followed by the lexicographic list of strictly increasing sequences on the symbols ij, $0 \leqq i < j < r$.

For example, there are

$$\left(\begin{matrix}\binom{10}{2}+10+4-1\\ 5\end{matrix}\right)+\left(\begin{matrix}\binom{10}{2}\\ 5\end{matrix}\right)=\binom{59}{5}+\binom{45}{5}$$

$$=5006386+1221759=6228145$$

colorations of the 5-cube by 10 colors. The 6000000th coloration has rank

$$N=5999999-5006386=993613$$

in the list of strictly increasing sequences of length 5 on the symbols ij, $0 \leqq i < j < 10$.

$$\binom{45}{5}-1-993613=\binom{44-a_5}{1}+\binom{44-a_4}{2}+\binom{44-a_3}{3}+\binom{44-a_2}{4}+\binom{44-a_1}{5},$$

$$228145=\binom{2}{1}+\binom{14}{2}+\binom{27}{3}+\binom{29}{4}+\binom{32}{5},$$

so $a_1a_2a_3a_4a_5 = 12 \quad 15 \quad 17 \quad 30 \quad 42$.

These are the ranks of the symbols

$$15 \quad 18 \quad 23 \quad 45 \quad 78$$

in the list of 45 symbols ij, $0 \leqq i < j < 10$. Hence the 6000000th coloration of the 5-cube by 10 colors is

$$51 \quad 18 \quad 23 \quad 45 \quad 78.$$

In the opposite direction, the ranks of the symbols

$$01 \quad 23 \quad 45 \quad 67 \quad 89$$

in the list of 45 symbols ij are

$$0 \quad 17 \quad 30 \quad 39 \quad 44,$$

so $a_1a_2a_3a_4a_5 = 0 \quad 17 \quad 30 \quad 39 \quad 44$;

$$\binom{45}{5}-1-N=\binom{44-44}{1}+\binom{44-39}{2}+\binom{44-30}{3}+\binom{44-17}{4}+\binom{44-0}{5},$$

$$1221759-1-N=\binom{0}{1}+\binom{5}{2}+\binom{14}{3}+\binom{27}{4}+\binom{44}{5}=1103932,$$

so $N = 117826$.

Hence

$$10 \quad 23 \quad 45 \quad 67 \quad 89$$

has rank

$$5006386+117826=5124212$$

and is the 5124213th coloration of the 5-cube by 10 colors.

The number of colorations of the n-cube up to these symmetries may also be obtained from an extension of Pólya's theory of counting; such a counting neither produces the structures nor ranks them.

REFERENCE

[1] D. H. LEHMER, *The machine tools of combinatorics*, Applied Combinatorial Mathematics, E. F. Beckenbach, ed., John Wiley, New York, 1964, pp. 5–31.